

# ON THE LANGLANDS RETRACTION

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ABSTRACT. Given a root system in a vector space  $V$ , Langlands defined in 1973 a canonical retraction  $\mathfrak{L} : V \rightarrow V^+$ , where  $V^+ \subset V$  is the dominant chamber. In this note we give a short review of the material on this retraction (which is well known under the name of “Langlands’ geometric lemmas”).

The main purpose of this review is to provide a convenient reference for the work [DrGa], in which the Langlands retraction is used to define a coarsening of the Harder-Narasimhan-Shatz stratification of the stack of  $G$ -bundles on a smooth projective curve.

## 1. INTRODUCTION

Given a root system in a Euclidean space  $V$ , Langlands defined in [La2, Sect. 4] a certain retraction  $\mathfrak{L} : V \rightarrow V^+$ , where  $V^+$  is the dominant chamber. Later this retraction was discussed in [BoWa, Ch. IV, Subsect. 3.3] and [C, Sect. 1].

In this note we briefly recall the definition and properties of  $\mathfrak{L}$ . It has no new results compared with [La2] and [C]; my goal is only to provide a convenient reference for the work [DrGa] and possibly for some future works.

Following J. Carmona, we begin in Sect. 2 with the most naive definition of  $\mathfrak{L}$  (which makes sense for a Euclidean space equipped with *any* basis  $\{\alpha_i\}$ ): namely,  $\mathfrak{L}(x)$  is the point of  $V^+$  closest to  $x$ .

Starting with Section 3, we assume that  $\langle \alpha_i, \alpha_j \rangle \leq 0$  for  $i \neq j$ . The key point is that under this assumption  $\mathfrak{L}$  can be characterized in terms of the usual ordering on  $V$ : namely, Corollary 3.2 says that  $\mathfrak{L}(x)$  is the least element of the set

$$(1.1) \quad \{y \in V^+ \mid y \geq x\}.$$

It is this characterization of  $\mathfrak{L}$  that is important for most applications (in particular, it is used in [DrGa, Appendix B]). One can consider it as a definition of  $\mathfrak{L}$  and Corollary 3.2 as a way to prove the existence of the least element of the set (1.1). In Section 4 we give *another* proof of this fact, which is independent of Sections 2-3; closely related to it are Remark 4.2 and Example 4.3.

In Section 5 we define the Langlands retraction as a map from the space of rational coweights of a reductive group to the dominant cone.

In Section 6 we make some historical remarks.

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## 2. THE RETRACTION DEFINED BY THE METRIC

Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$  with a positive definite scalar product  $\langle \cdot, \cdot \rangle$ . Let  $\{\alpha_i\}_{i \in \Gamma}$  be an arbitrary basis in  $V$  and  $\{\omega_i\}_{i \in \Gamma}$  the dual basis. Let  $V^+ \subset V$  denote the closed convex cone generated by the  $\omega_i$ ’s,  $i \in \Gamma$ .

Following J. Carmona [C, Sect. 1], we define the *Langlands retraction*  $\mathfrak{L} : V \rightarrow V^+$  as follows:  $\mathfrak{L}(x)$  is the point of  $V^+$  closest to  $x$  (such point exists and is unique because  $V^+$  is closed and convex). It is easy to see that the map  $\mathfrak{L}$  is continuous.

Let us give another description of  $\mathfrak{L}$ . For a subset  $J \subset \Gamma$  let  $K_J$  denote the closed convex cone generated by  $\omega_j$  for  $j \in \Gamma - J$  and by  $-\alpha_i$  for  $i \in J$ . Clearly, each  $K_J$  is a simplicial cone of full dimension in  $V$ . Let  $V_J$  denote the linear span of  $\alpha_j$ ,  $j \in J$  (so  $V_J^\perp$  is spanned by  $\omega_i$ ,  $i \notin J$ ). Let  $\text{pr}_J : V \rightarrow V$  denote the orthogonal projection onto  $V_J^\perp$ , so  $\ker(\text{pr}_J) = V_J$ .

**Proposition 2.1.** (a) *The map  $\mathfrak{L}$  is piecewise linear. The cones  $K_J$  are exactly the linearity domains of  $\mathfrak{L}$ . For  $x \in K_J$  one has  $\mathfrak{L}(x) = \text{pr}_J(x)$ .*  
(b) *The cones  $K_J$  and their faces form a complete simplicial fan<sup>1</sup> in  $V$ , combinatorially equivalent to the coordinate fan<sup>2</sup>.*

*Remark 2.2.* The wording in the above proposition was suggested to us by A. Zelevinsky.

The proposition immediately follows from the next lemma, whose proof is straightforward.

**Lemma 2.3.** *Let  $x \in V$  and  $y \in V^+$ . Set  $J := \{j \in \Gamma \mid \langle \alpha_j, y \rangle = 0\}$ . Then the following are equivalent:*

- (a)  $y = \mathfrak{L}(x)$ .
- (b)  $x - y$  belongs to the closed convex cone generated by  $-\alpha_j$  for  $j \in J$ . □

### 3. THE KEY STATEMENTS

Let  $V^{pos}$  denote the cone dual to  $V^+$ , i.e., the closed convex cone generated by the  $\alpha_i$ 's,  $i \in \Gamma$ . Equip  $V$  and  $V^+$  with the following partial ordering:  $x \leq y$  if  $y - x \in V^{pos}$ . By Lemma 2.3, the retraction  $\mathfrak{L} : V \rightarrow V^+$  from Section 2 has the following property:

$$(3.1) \quad \mathfrak{L}(x) \geq x, \quad x \in V.$$

**Theorem 3.1.** *Assume that*

$$(3.2) \quad \langle \alpha_i, \alpha_j \rangle \leq 0 \text{ for } i \neq j.$$

*Then the retraction  $\mathfrak{L} : V \rightarrow V^+$  is order-preserving.*

By (3.1), Theorem 3.1 implies the following statement, which characterizes  $\mathfrak{L}$  in terms of the order relation.

**Corollary 3.2.** *If (3.2) holds then  $\mathfrak{L}(x)$  is the least element in  $\{y \in V^+ \mid y \geq x\}$ . □*

Let us prove Theorem 3.1. To show that a piecewise linear map is order-preserving it suffices to check that this is true on each of its linearity domains. So Theorem 3.1 follows from Proposition 2.1(a) and the next proposition, which I learned from S. Schieder [Sch, Prop.3.1.2(a)].

**Proposition 3.3.** *Assume (3.2). Then for each subset  $J \subset \Gamma$  the map  $\text{pr}_J : V \rightarrow V$  defined in Section 2 is order-preserving.*

To prove the proposition, we need the following lemma.

**Lemma 3.4.** *Let  $J \subset \Gamma$ . Suppose that  $x \in V_J$  and  $\langle x, \alpha_j \rangle \geq 0$  for all  $j \in J$ . Then  $x \geq 0$ .*

*Proof of the lemma.* We can assume that  $J = \Gamma$  (otherwise replace  $V$  by  $V_J$  and  $\Gamma$  by  $\Gamma_J$ ). Then the lemma just says that  $V^+ \subset V^{pos}$ . This is a well known consequence of (3.2). □

<sup>1</sup>This means that these cones cover  $V$  and each intersection  $K_J \cap K_{J'}$  is a face in both  $K_J$  and  $K_{J'}$ .

<sup>2</sup>The coordinate fan is what one gets when the basis  $\{\alpha_i\}$  is orthogonal.

*Proof of Proposition 3.3.* We have to show that  $\text{pr}_J(\alpha_i) \geq 0$  for any  $i \in \Gamma$ . If  $i \in J$  then  $\text{pr}_J(\alpha_i) = 0$ . Now suppose that  $i \notin J$ . By the definition of  $\text{pr}_J$ , we have  $\text{pr}_J(\alpha_i) = \alpha_i + x$ , where  $x$  is the element of  $V_J$  such that  $\langle x, \alpha_j \rangle = -\langle \alpha_i, \alpha_j \rangle$  for all  $j \in J$ . By (3.2) and Lemma 3.4,  $x \geq 0$ , so  $\text{pr}_J(\alpha_i) = \alpha_i + x \geq 0$ .  $\square$

#### 4. ANOTHER APPROACH TO THE LANGLANDS RETRACTION

Suppose that (3.2) holds. Then one could take Corollary 3.2 as the *definition* of the Langlands retraction  $\mathfrak{L} : V \rightarrow V^+$ , i.e., one could define  $\mathfrak{L}(x)$  to be the least element of the set  $\{y \in V^+ \mid y \geq x\}$ . This set is closed and non-empty (because (3.2) implies that  $V^+ \subset V^{\text{pos}}$ ), so the existence of the least element in it follows from the next proposition.

**Proposition 4.1.** *Suppose that  $\langle \alpha_i, \alpha_j \rangle \leq 0$  for  $i \neq j$ . Then the infimum of any non-empty subset of  $V^+$  belongs to  $V^+$ .*

Here “infimum” is understood in terms of the partial ordering defined by  $V^{\text{pos}}$ . In other words, given a family of vectors

$$(4.1) \quad x_t \in V, \quad x_t = \sum_i x_{i,t} \cdot \alpha_i,$$

its infimum equals  $\sum_i y_i \cdot \alpha_i$ , where  $y_i := \inf_t x_{i,t}$ . Note that if  $x_t \in V^+$  then  $x_t \in V^{\text{pos}}$ , so  $x_{i,t} \geq 0$  and  $\inf_t x_{i,t}$  exists.

*Proof of Proposition 4.1.* Suppose that we have a family of vectors  $x_t \in V^+$  and  $y = \inf_t x_t$ . The assumption  $x_t \in V^+$  means that  $\langle x_t, \alpha_i \rangle \geq 0$  for all  $i$ . We have to show that  $\langle y, \alpha_i \rangle \geq 0$  for all  $i$ .

Fix  $i$ . Write  $x_t = x'_t + x''_t$ ,  $y = y' + y''$ , where

$$x'_t, y' \in \mathbb{R}\alpha_i, \quad x''_t, y'' \in \bigoplus_{j \neq i} \mathbb{R}\alpha_j.$$

Clearly  $y' = \inf_t x'_t$ ,  $y'' = \inf_t x''_t$ . Then for every  $t$  one has

$$\langle x'_t, \alpha_i \rangle = \langle x_t, \alpha_i \rangle - \langle x''_t, \alpha_i \rangle \geq -\langle x''_t, \alpha_i \rangle \geq -\langle y'', \alpha_i \rangle$$

(the second inequality holds because  $-\langle \alpha_j, \alpha_i \rangle \geq 0$  for  $j \neq i$ ). So

$$\langle y', \alpha_i \rangle = \inf_t \langle x'_t, \alpha_i \rangle \geq -\langle y'', \alpha_i \rangle,$$

i.e.,  $\langle y, \alpha_i \rangle \geq 0$ .  $\square$

*Remark 4.2.* In the situation of the following example Proposition 4.1 just says that the infimum of any family of concave functions is concave. In fact, the above proof of Proposition 4.1 is identical to the proof of this classical statement.

*Example 4.3.* Consider the root system of  $SL(n)$ . In this case  $V$  is the orthogonal complement of the vector  $\varepsilon_1 + \dots + \varepsilon_n$  in the Euclidean space with orthonormal basis  $\varepsilon_1, \dots, \varepsilon_n$ , and  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ,  $1 \leq i \leq n-1$ . Let  $\omega_i \in V$  be the basis dual to  $\alpha_i$ . For each  $v \in V$  define  $f_v : \{0, \dots, n\} \rightarrow \mathbb{R}$  by

$$f_v(0) = f_v(n) = 0, \quad f_v(i) = \langle v, \omega_i \rangle \quad \text{for } 0 < i < n.$$

Then the map  $v \mapsto f_v$  identifies  $V$  with the space of functions  $f : \{0, \dots, n\} \rightarrow \mathbb{R}$  such that  $f(0) = f(n) = 0$ . Moreover,  $V^{\text{pos}}$  identifies with the subset of non-negative functions  $f$  and  $V^+$  with the subset of *concave* functions  $f$ . Thus the Langlands retraction assigns to a function  $f : \{0, \dots, n\} \rightarrow \mathbb{R}$  the smallest concave function which is  $\geq f$ .

## 5. REDUCTIVE GROUPS

**5.1. A remark on rationality.** Suppose that in the situation of Sect. 2 one has  $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Q}$  for all  $i, j \in \Gamma$ . Then the  $\mathbb{Q}$ -linear span of the  $\alpha'_i$ 's equals the  $\mathbb{Q}$ -linear span of the  $\omega'_i$ 's. Denote it by  $V^{\mathbb{Q}}$ . Then  $V = V^{\mathbb{Q}} \otimes \mathbb{R}$ . The cones  $K_J$ , the subspaces  $V_J$ , and the operators  $\text{pr}_J$  from Section 2 are clearly defined over  $\mathbb{Q}$ . So by Proposition 2.1, one has

$$(5.1) \quad \mathfrak{L}(V^{\mathbb{Q}}) \subset V^{\mathbb{Q}}.$$

**5.2. The Langlands retraction for coweights.** Now let  $G$  be a connected reductive group over an algebraically closed field. Let  $\Lambda_G$  be its coweight lattice, i.e.,  $\Lambda_G = \text{Hom}(\mathbb{G}_m, T)$ , where  $T$  is the maximal torus of  $G$ . Set  $\Lambda_G^{\mathbb{Q}} := \Lambda_G \otimes \mathbb{Q}$ . We have the simple coroots  $\check{\alpha}_i \in \Lambda_G$  and the simple roots  $\alpha_i \in \text{Hom}(\Lambda_G, \mathbb{Z})$ . Let  $\Lambda_G^{+, \mathbb{Q}} \subset \Lambda_G^{\mathbb{Q}}$  denote the dominant cone. Equip  $\Lambda_G^{+, \mathbb{Q}}$  with the following partial ordering:  $\lambda_1 \leq \lambda_2$  if  $\lambda_2 - \lambda_1$  is a linear combination of the simple coroots with non-negative coefficients.

Now define the *Langlands retraction*  $\mathfrak{L}_G : \Lambda_G^{\mathbb{Q}} \rightarrow \Lambda_G^{+, \mathbb{Q}}$  as follows:  $\mathfrak{L}_G(\lambda)$  is the least element of the set

$$(5.2) \quad \{\mu \in \Lambda_G^{+, \mathbb{Q}} \mid \mu \geq \lambda\}$$

with respect to the  $\leq$  ordering.

**Corollary 5.3.** (i)  $\mathfrak{L}_G(\lambda)$  exists.

(ii)  $\mathfrak{L}_G(\lambda)$  is the element of  $\Lambda_G^{+, \mathbb{Q}}$  closest to  $\lambda$  with respect to any positive scalar product on  $\Lambda_G^{+, \mathbb{Q}} \otimes \mathbb{R}$  which is invariant with respect to the Weyl group.

(iii)  $\mathfrak{L}_G(\lambda)$  is the unique element of the set (5.2) with the following property:  $\langle \mathfrak{L}_G(\lambda), \alpha_i \rangle = 0$  for any simple root  $\alpha_i$  such that the coefficient of  $\check{\alpha}_i$  in  $\mathfrak{L}_G(\lambda) - \lambda$  is nonzero.

*Proof.* Combine Lemma 2.3, Corollary 3.2, and the inclusion (5.1).  $\square$

**5.4. Example:**  $G = GL(n)$ . In this case, just as in Example 4.3, one identifies  $\Lambda_G^{\mathbb{Q}}$  with the space of functions  $f : \{0, \dots, n\} \rightarrow \mathbb{Q}$  such that  $f(0) = 0$  (while  $f(n)$  is arbitrary). Then the subset  $\Lambda_G^{\mathbb{Q}} \subset \Lambda_G^{+, \mathbb{Q}}$  identifies with the subset of *concave* functions  $f : \{0, \dots, n\} \rightarrow \mathbb{Q}$  with  $f(0) = 0$ . Just as in Example 4.3, the Langlands retraction assigns to a function  $f : \{0, \dots, n\} \rightarrow \mathbb{Q}$  the smallest concave function which is  $\geq f$ .

## 6. SOME HISTORICAL REMARKS

In [La2] R. Langlands defined the retraction  $\mathfrak{L}$  and formulated his “geometric lemmas” (see [La2, Lemmas 4.4-4.5 and Corollary 4.6]) for the purpose of the classification of representations of real reductive groups in terms of tempered ones. However, much earlier he had formulated a closely related (and more complicated) combinatorial lemma<sup>3</sup> in his theory of Eisenstein series, see [La1, Sect. 8]. In this work Langlands considers Eisenstein series on quotients of the form  $G(\mathbb{R})/\Gamma$ , where  $G$  is a reductive group over  $\mathbb{Q}$  and  $\Gamma$  is an arithmetic subgroup, but the same technique applies to quotients of the form  $G(\mathbb{A})/G(\mathbb{Q})$ . Note that the stack  $\text{Bun}_G$  considered in [DrGa] is not far away from  $G(\mathbb{A})/G(\mathbb{Q})$ , so the fact that the Langlands retraction is used in [DrGa, Appendix B] is not surprising.

<sup>3</sup>An elementary introduction to this lemma can be found in [Cas1, Cas2].

## REFERENCES

- [BoWa] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Mathematical Surveys and Monographs **67**, AMS, (2000).
- [C] J. Carmona, *Sur la classification des modules admissibles irréductibles*, in: Noncommutative harmonic analysis and Lie groups (Marseille, 1982), 11–34, Lecture Notes in Math., **1020**, Springer-Verlag, Berlin, 1983.
- [Cas1] W. Casselman, *Truncation exercises*, in: Functional analysis VIII, 84–104, Various Publ. Ser. (Aarhus), **47**, Aarhus Univ., Aarhus, 2004; also available at <http://www.math.ubc.ca/~cass/research.html>
- [Cas2] W. Casselman, *Chipping away at convex sets* (slides of a 2004 talk in Sydney), available at <http://www.math.ubc.ca/~cass/sydney/>
- [DrGa] V. Drinfeld and D. Gaietsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv:1112.2402.
- [La1] R. P. Langlands, *Eisenstein series*, in: Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), 235–252, Amer. Math. Soc., Providence, R.I., 1966.
- [La2] R. Langlands, *On the classification of irreducible representations of real algebraic groups*, in: Representation theory and harmonic analysis on semi-simple Lie groups, edited by P. Sally and D. Vogan, AMS 1989, 101–170.
- [Sch] S. Schieder, *On the Harder-Narasimhan stratification of  $\mathrm{Bun}_G$  via Drinfeld's compactifications*, arXiv:1212.6814.

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